https://www.linkedin.com/feed/update/urn:li:activity:6617828535992229888

Find $404\left((x+y)^2-2\right)^2$, if $x>0$ and $y>0$ such that

$$x=y+\cfrac{1}{x+\cfrac{1}{y+\cfrac{1}{x+\dots}}},\quad y=x-\cfrac{1}{y+\cfrac{1}{x-\cfrac{1}{y+\dots}}}.$$

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Since from given equations follows that infinite continued fractions in the right hand of its sides are convergent then both of these equations can be equivalentely rewritten as $x=y+\dfrac{1}{x+\frac{1}{x}}$ and $y=x-\dfrac{1}{y+\frac{1}{y}}$, respectively.

Thus, we have $x=y+\dfrac{1}{x+\frac{1}{x}} \Leftrightarrow x=y+\dfrac{x}{x^2+1} \Leftrightarrow x^3+x=x^2y+y+x \Leftrightarrow$ $x^3=x^2y+y \Leftrightarrow y=\dfrac{x^3}{x^2+1}$ and $y=x-\dfrac{1}{y+\frac{1}{y}} \Leftrightarrow y=x-\dfrac{y}{y^2+1} \Leftrightarrow$ $y^3+y=xy^2+x-y \Leftrightarrow y^3=xy^2+x-2y$. Hence, $x^3+y^3=(x^2y+y)+(xy^2+x-2y) \Leftrightarrow$ $x^3+y^3-(x^2y+xy^2)=x-y \Leftrightarrow (x+y)(x-y)^2=x-y$.

Noting that $x\neq y$ (otherwise equation $y=\dfrac{x^3}{x^2+1}$ becomes $x=\dfrac{x^3}{x^2+1} \Leftrightarrow x=0$) we obtain $(x+y)(x-y)^2=x-y \Leftrightarrow x^2-y^2=1$.

Then by substitution $y=\dfrac{x^3}{x^2+1}$ in the latter equation we obtain

$x^2-\left(\dfrac{x^3}{x^2+1}\right)^2=1 \Leftrightarrow x^4-x^2-1=0 \Leftrightarrow x^2=\dfrac{1+\sqrt{5}}{2}$ and, therefore,

$y^2=x^2-1=\dfrac{1+\sqrt{5}}{2}-1=\dfrac{\sqrt{5}-1}{2}$. Hence, $(x+y)^2-2=x^2+y^2+2xy-2=$ $\dfrac{\sqrt{5}+1}{2}+\dfrac{\sqrt{5}-1}{2}+2\cdot\dfrac{\sqrt{5}+1}{2}\cdot\dfrac{\sqrt{5}-1}{2}-2=\sqrt{5}$ and we obtain

$404\left((x+y)^2-2\right)^2=404\cdot 5=2020.$